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Lowest distortion mappings

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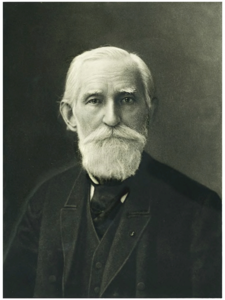
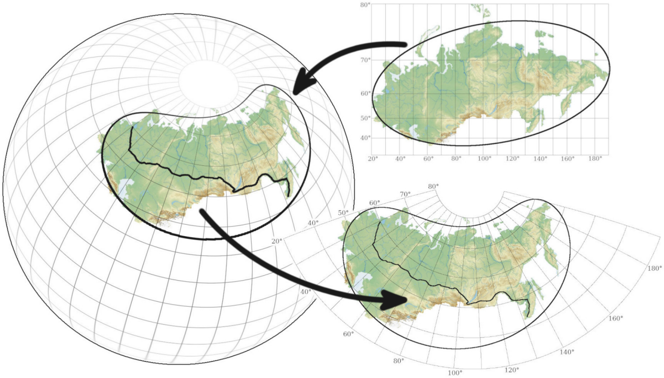
François Protais, Dmitry Sokolov

Tetrahedron VII, October 10, 2023

Chebyshev problem: best quasi-isometric mapping





P.L. Chebyshev, 1856:

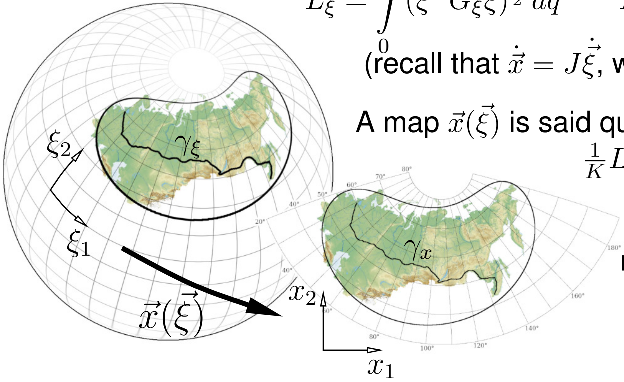
- Map of the country is optimal when it provides minimal relative path length error
- Conformal mapping is optimal if isometric on the boundary of the domain
- The relative length error of TransSib (St.Petersburg - Vladivostok railway) is less than 1.2% (103 kilometers for about 9000 km total length).

Quasi-isometric mapping: formal definition

Consider a map $\vec{x}(\vec{\xi}) : \Omega \to \Omega_x$, where $\Omega, \Omega_x \subset \mathbb{R}^d$ with corresponding metric tensors $G_\xi(\vec{\xi})$ and $G_x(\vec{x})$. For any simple curve $\gamma_\xi \in \Omega$ defined by 1D parameterization $\vec{\xi}(q), 0 \le q \le Q$, and its image γ_x we can measure their lengths as:

$$L_\xi = \int_0^Q (\dot{\vec{\xi}}^\top G_\xi \dot{\vec{\xi}})^{\frac{1}{2}} \, dq \qquad L_x = \int_0^Q (\dot{\vec{\xi}}^\top J^\top G_x J \dot{\vec{\xi}})^{\frac{1}{2}} \, dq$$

(recall that $\dot{\vec{x}} = J\dot{\vec{\xi}}$, where J is the Jacobian matrix)

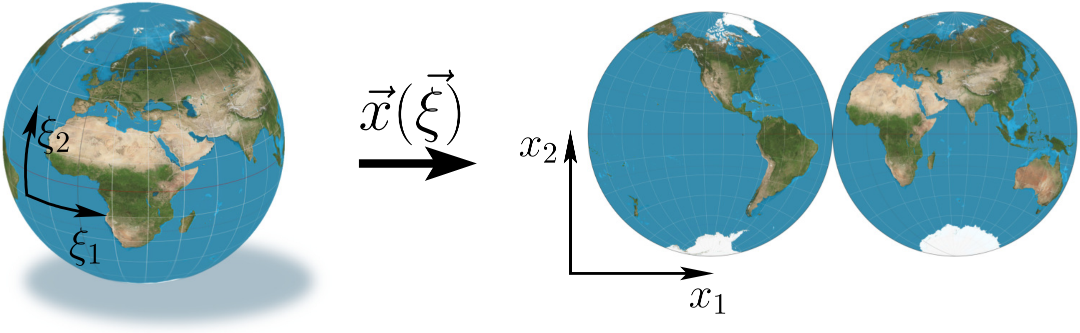


A map $\vec{x}(\vec{\xi})$ is said quasi-isometric if there exists a bound K:
$\frac{1}{K} L_\xi < L_x < K L_\xi$ for any curve γ_ξ.

For a map regular enough we can reformulate it as a local matrix inequality:

$$\frac{1}{K^2} G_\xi < J^\top G_x J < K^2 G_\xi.$$

Elasticity as proxy problem



How to compute the best $\vec{x}(\vec{\xi})$? Matrix-based optimization seems hard...

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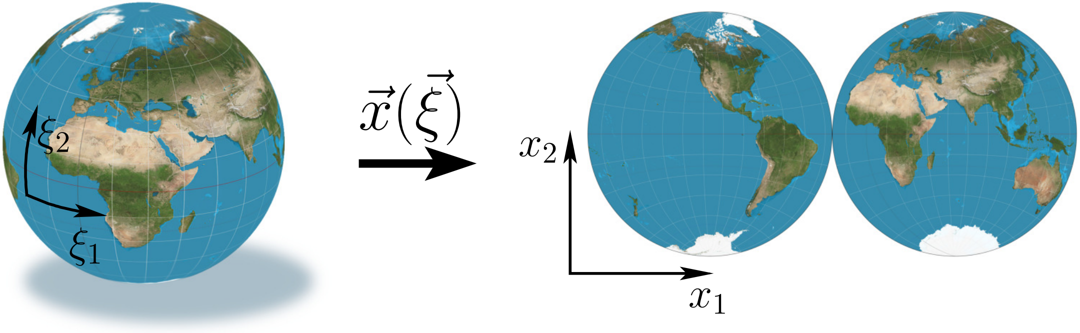

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💡 Solve a proxy problem instead: mathematical elasticity minimizes the deviation from the isometric deformation state on average:

$\min\limits_{\vec{x}(\vec{\xi})} \int\limits_{\Omega} f(J)\, d\xi$, where f is the distortion measure and J is the Jacobian matrix.

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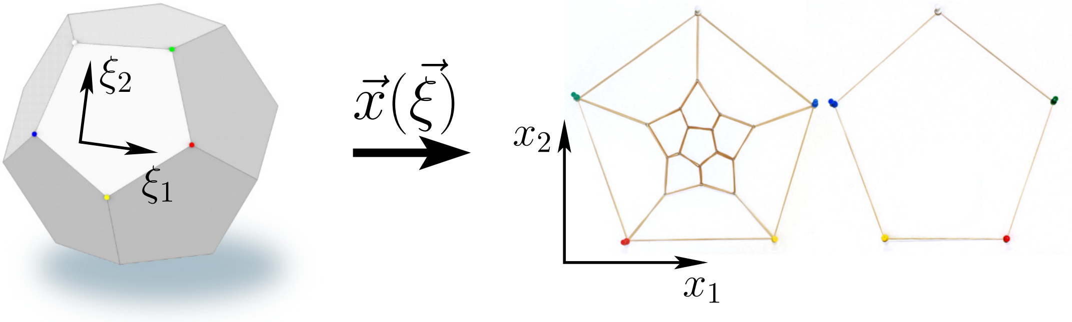

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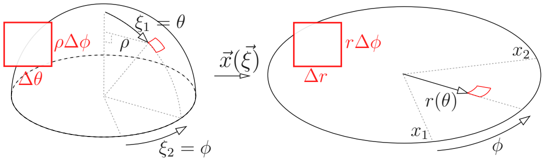
G.B. Airy, 1861: Balance-of-Errors



LIII. *Explanation of a Projection by Balance of Errors for Maps applying to a very large extent of the Earth's Surface; and Comparison of this projection with other projections.* By G. B. Airy, *Esq., Astronomer Royal**.

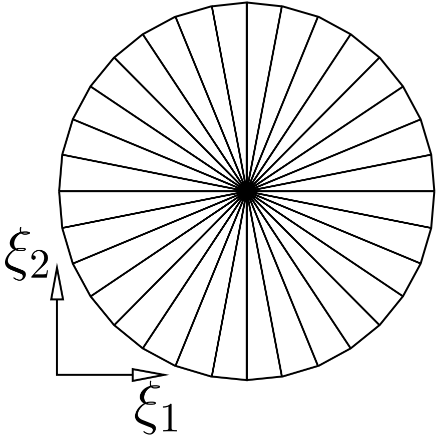
3. The two errors, to one or both of which all projections are liable, are, Change of Area, and Distortion, as applying to small portions of the earth's surface. On the one hand, a projection may be invented (to which I shall give the name of "Projection with Unchanged Areas") in which there is no Change of Area, but excessive Distortion, for parts far from the centre; on the other hand, the Stereographic Projection has no Distortion, but has great Change of Area for distant parts. Between these lie the projections which have usually been adopted by geographers, with the tacit purpose of greatly reducing the error of one kind by the admission of a small error of the other kind, but without any distinctly-expressed principle (so far as I know) for their guidance in the details of the projection.

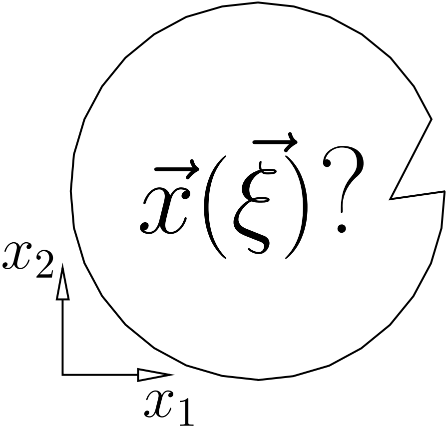
4. My object in this paper is to exhibit a distinct mathematical process for determining the magnitudes of these errors, so that the result of their combination shall be most advantageous. This principle I call "The Balance of Errors." It is founded upon the following assumptions and inferences:—

Consequently the summation of the partial evils for the whole map is represented by

$$\int d\theta . \left\{ \left(\frac{dr}{d\theta} - 1\right)^2 + \left(\frac{r}{\sin\theta} - 1\right)^2 \right\} \times \sin\theta.$$

Toy example: average vs max distortion





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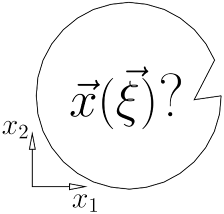
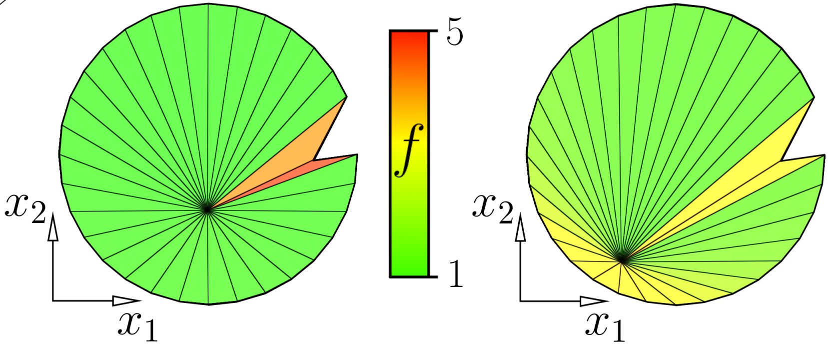


$$\min_{\vec{x}(\vec{\xi})} \int_{\Omega} f(J)\, d\xi$$

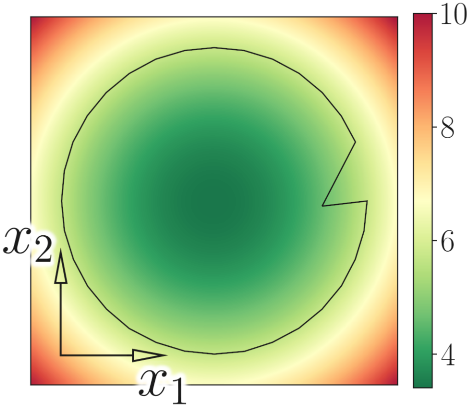
$$\min_{\vec{x}(\vec{\xi})} \max_{\vec{\xi}} f(J)$$



Barrier functional (Ivanenko, 1988)

Dirichlet ($\Delta\vec{x}(\vec{\xi}) = \vec{0}$):

$$\min \int_\Omega f(J)d\xi, \quad f(J) := \frac{1}{2}\|J\|_F^2$$



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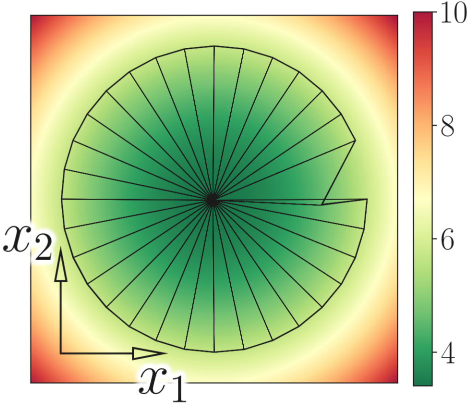
⚠ **Inverted elements!**



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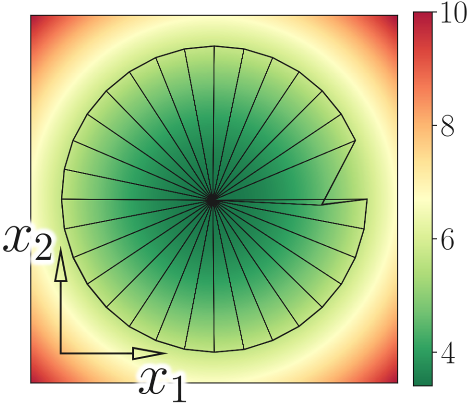

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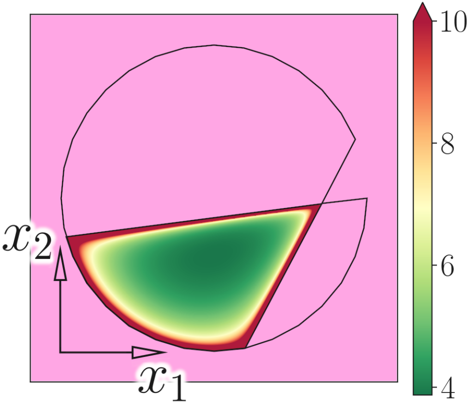

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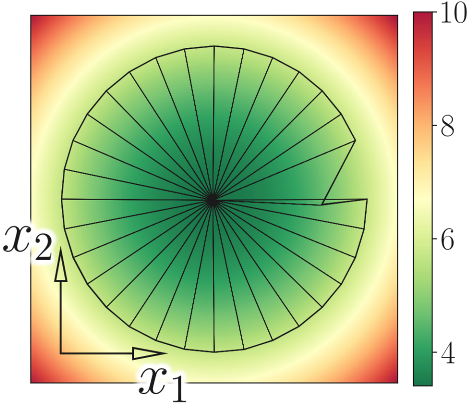

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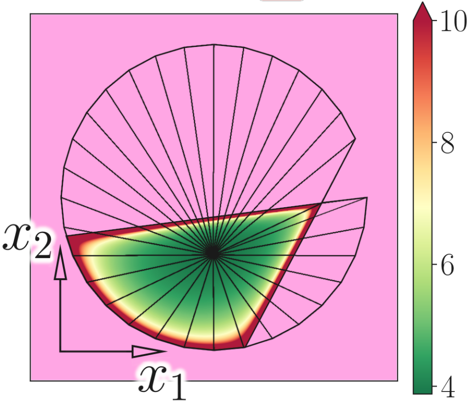
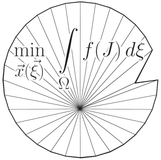
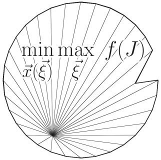
No inversions, but ⚠ **how to solve?**



Bounded distortion: stiffening (Garanzha, 2000)

$$\min \int\limits_{\Omega} w_t(J) \cdot f(J) d\xi$$





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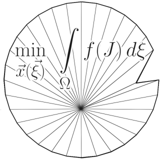

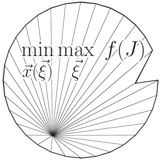

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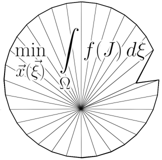

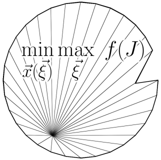

$f(J) \geq 1$, so if the integral is finite,
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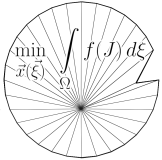

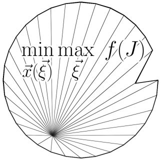

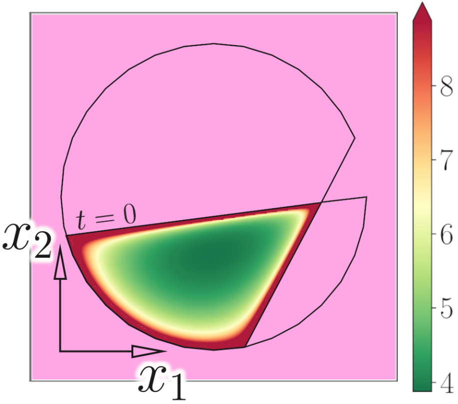

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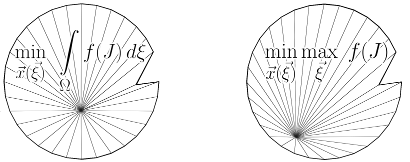

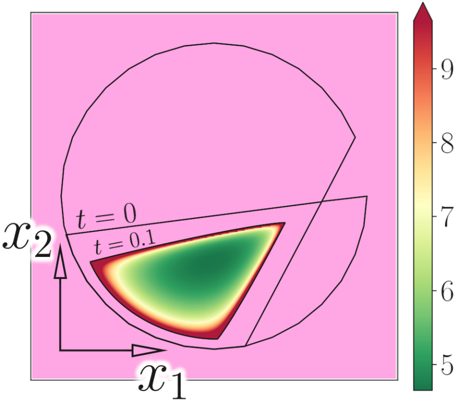

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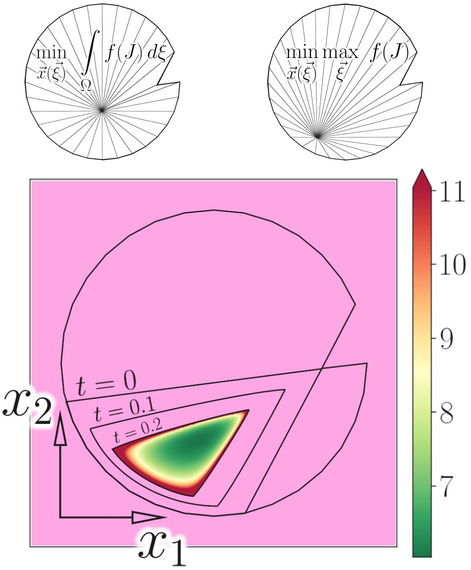

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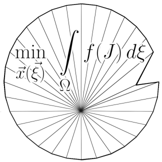

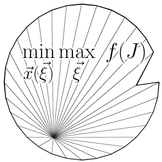

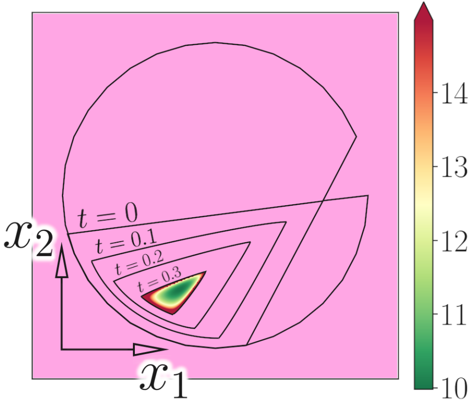

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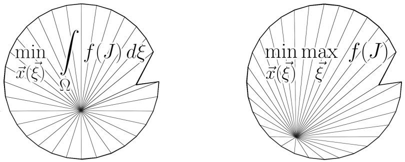

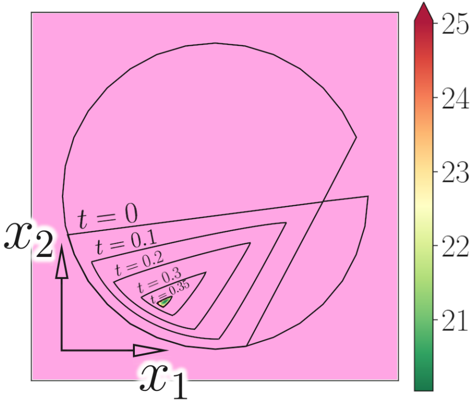

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- How to choose t?
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Untangling (Garanzha 1999)

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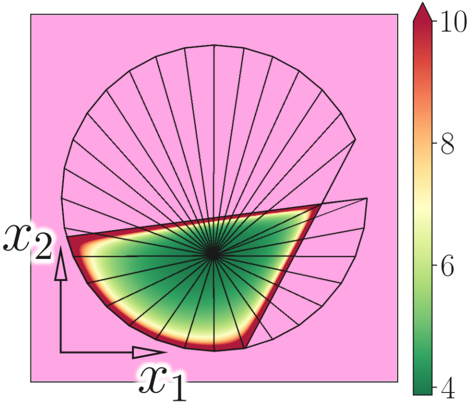

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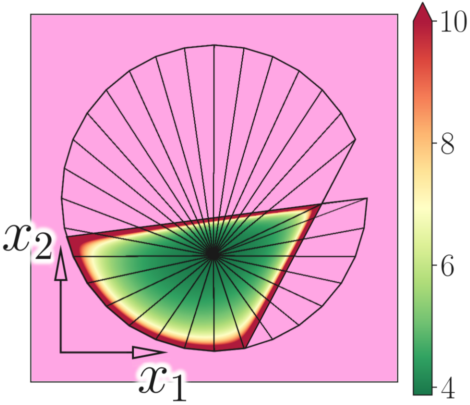
How to solve?



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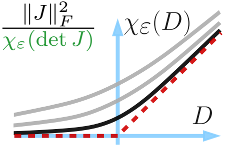

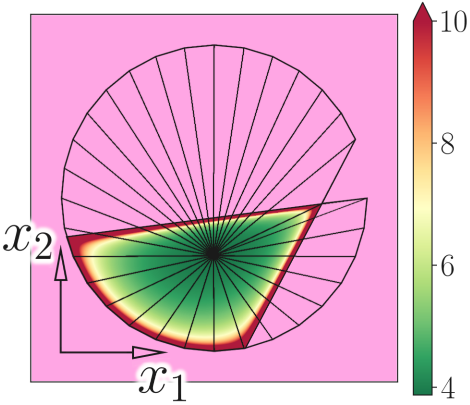

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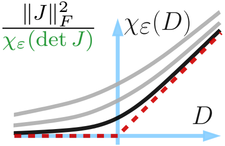

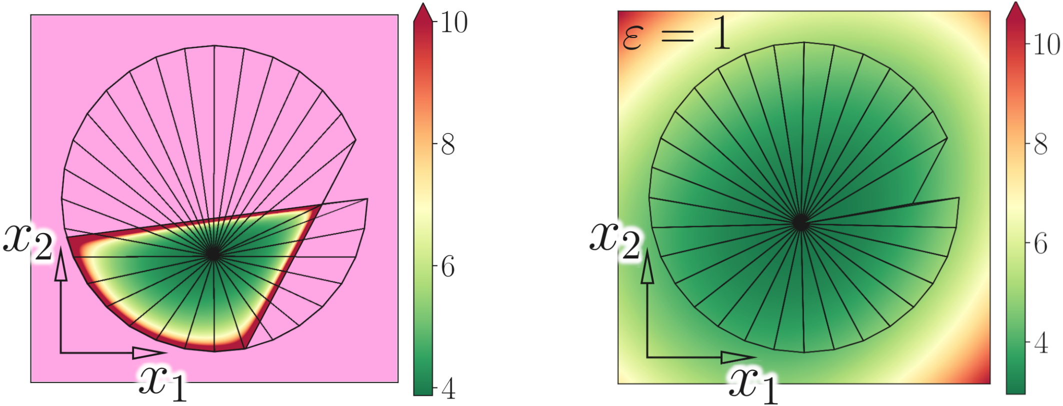

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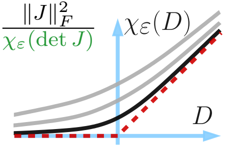

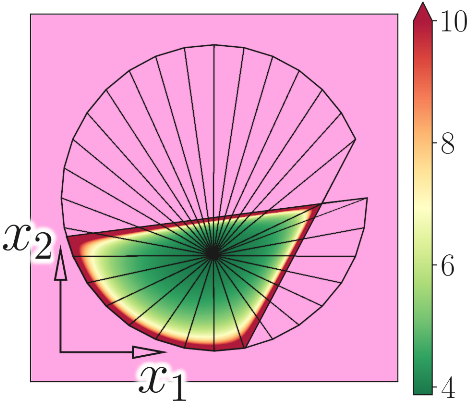

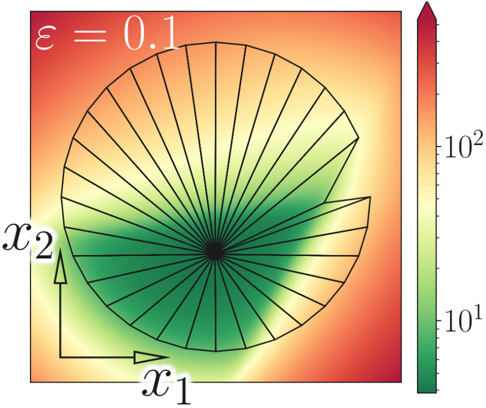

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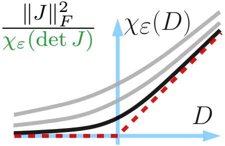

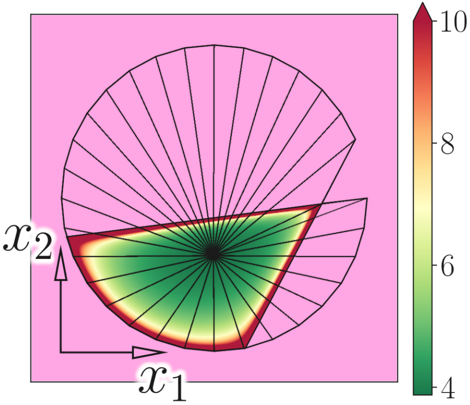

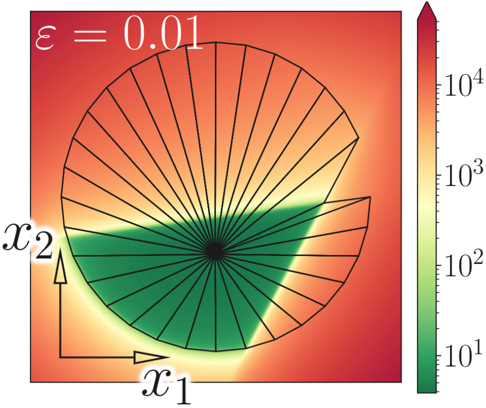

Foldover-free vs lowest distortion mapping

Build a decreasing sequence of ε^k:

Untangling algorithm

Input: $\vec{x}^0(\vec{\xi})$ *// arbitrary initial guess*

Output: $\vec{x}^*(\vec{\xi})$ *// foldover-free map*

$k \leftarrow 0$;

repeat

  compute ε^k;

  $\vec{x}^{k+1}(\vec{\xi}) \leftarrow \underset{\vec{x}(\vec{\xi})}{\arg\min} \int\limits_{\Omega} f_{\varepsilon^k}(J)d\xi$;

  $k \leftarrow k+1$;

until $\min\limits_{\Omega} \det J > 0$ **and convergence**

$\vec{x}^*(\vec{\xi}) \leftarrow \vec{x}^k(\vec{\xi})$;

Foldover-free vs lowest distortion mapping

Build a decreasing sequence of ε^k:

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$k \leftarrow 0;$
repeat
  compute ε^k;
  $\vec{x}^{k+1}(\vec{\xi}) \leftarrow \underset{\vec{x}(\vec{\xi})}{\arg\min} \int_\Omega f_{\varepsilon^k}(J)d\xi;$
  $k \leftarrow k+1;$
until $\min_\Omega \det J > 0$ **and convergence**
$\vec{x}^*(\vec{\xi}) \leftarrow \vec{x}^k(\vec{\xi});$

Build an increasing sequence of t^k:

Stiffening algorithm

Input: $\vec{x}^0(\vec{\xi})$ *// foldover-free initial guess*
Output: $\vec{x}^*(\vec{\xi})$ *// lowest-distortion map*

$k \leftarrow 0;$
repeat
  compute t^k;
  $\vec{x}^{k+1}(\vec{\xi}) \leftarrow \underset{\vec{x}(\vec{\xi})}{\arg\min} \int_\Omega \frac{f(J)}{\max(0, 1-t^k \cdot f(J))}d\xi;$
  $k \leftarrow k+1;$
until convergence
$\vec{x}^*(\vec{\xi}) \leftarrow \vec{x}^k(\vec{\xi});$

“Finite number of steps” theorems

💡 Build a decreasing sequence of ε^k: [Garanzha et al 2021]
Foldover-Free Maps in 50 Lines of Code

Untangling in a finite number of steps

If we have an efficient solver* and the admissible set is not empty,

then it is reachable by solving a finite number of minimization problems.

* *Few counter-examples were found.*

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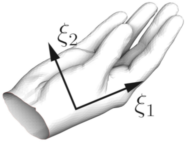
💡 Build an increasing sequence of t^k:
[Garanzha et al 2023]
In the Quest for Scale-Optimal Mappings

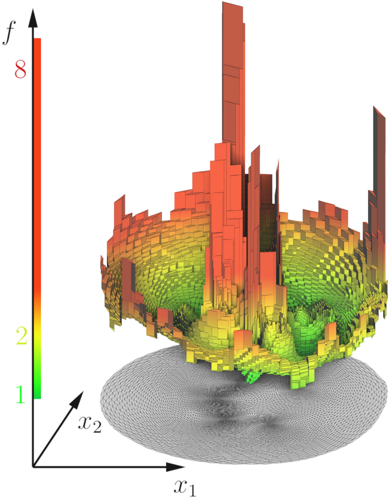
Bounded distortion in a finite number of steps

If we have an efficient solver** and the admissible set is not empty for a given distortion bound,
then it is reachable by solving a finite number of minimization problems.

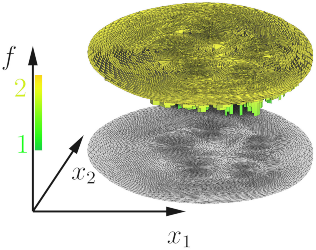
** *Counter-examples are not (yet?) exposed.*

Free-boundary flattening example



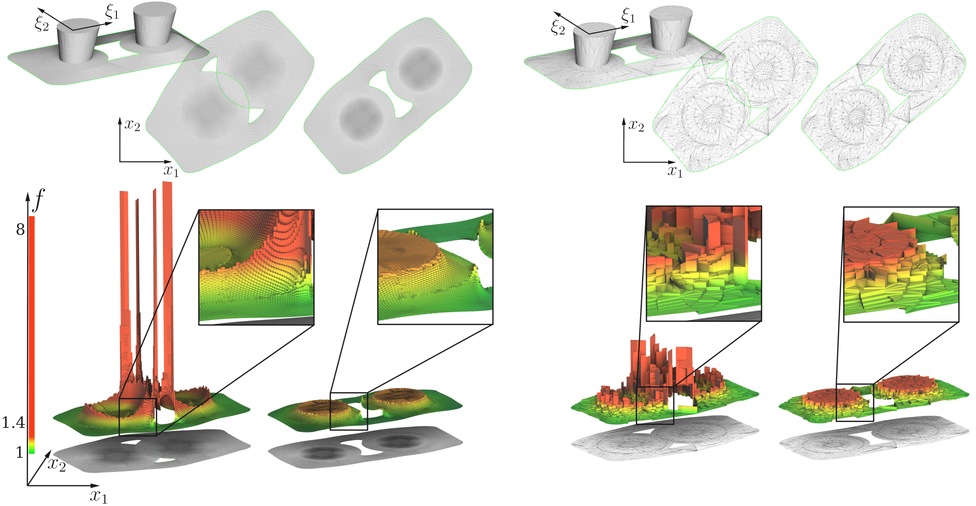


$$\min_{\vec{x}(\vec{\xi})} \int_{\Omega} f(J) d\xi$$



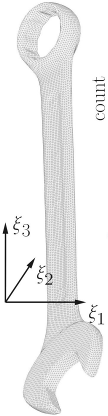
$$\min_{\vec{x}(\vec{\xi})} \max_{\Omega} f(J)$$

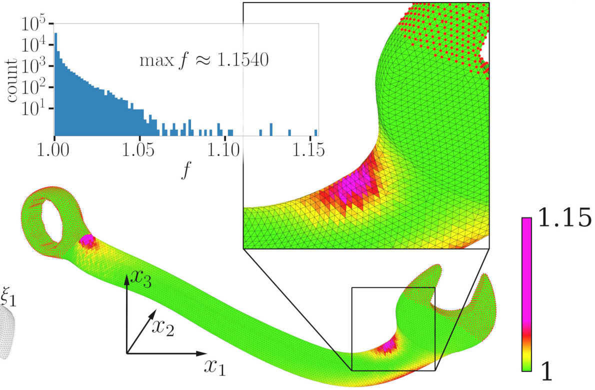
Free-boundary flattening stress test



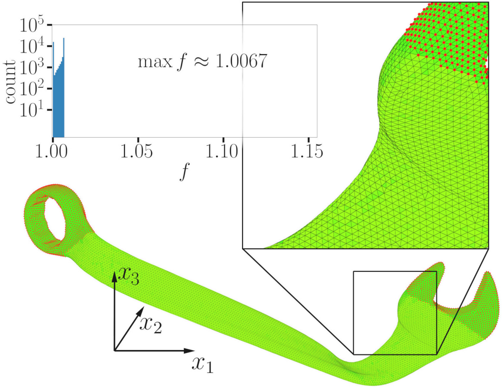
$\min_{\vec{x}(\vec{\xi})} \int_{\Omega} f(J)d\xi$ $\min_{\vec{x}(\vec{\xi})} \max_{\Omega} f(J)$ $\min_{\vec{x}(\vec{\xi})} \int_{\Omega} f(J)d\xi$ $\min_{\vec{x}(\vec{\xi})} \max_{\Omega} f(J)$

3D deformation example



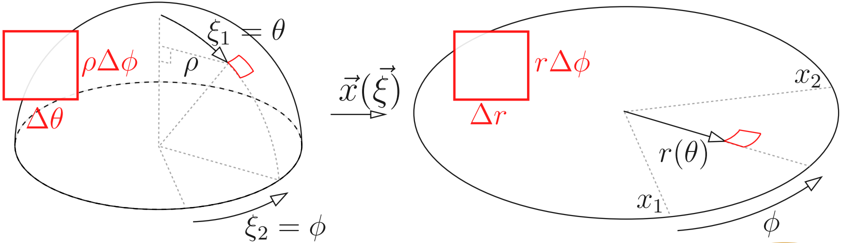


$$\min_{\vec{x}(\vec{\xi})} \int_\Omega f(J) d\xi$$



$$\min_{\vec{x}(\vec{\xi})} \max_\Omega f(J)$$

Flat Earth mathematics



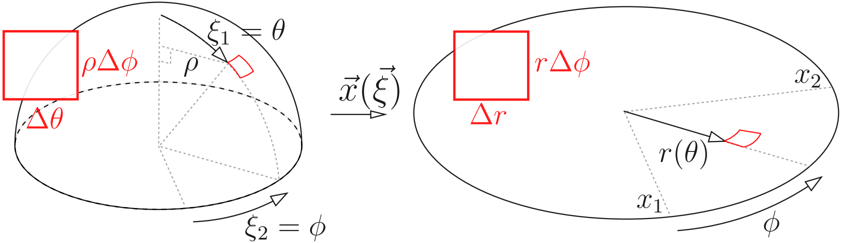
Ground truth

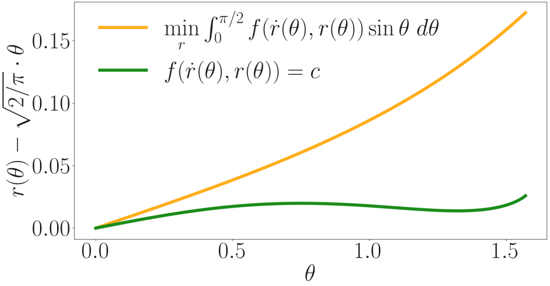
Azimuthal equidistant projection is scale-optimal:

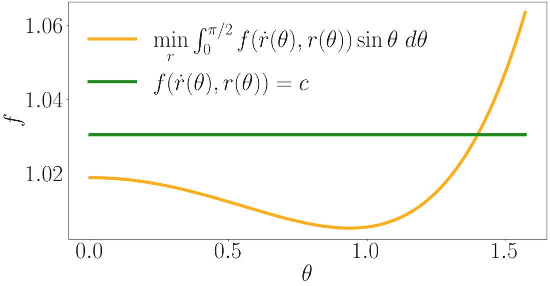
$$r(\theta) = \sqrt{\frac{2}{\pi}}\theta$$

(see [Milnor 1969] for the proof)

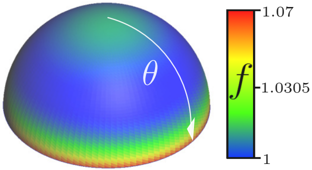
Flat Earth mathematics

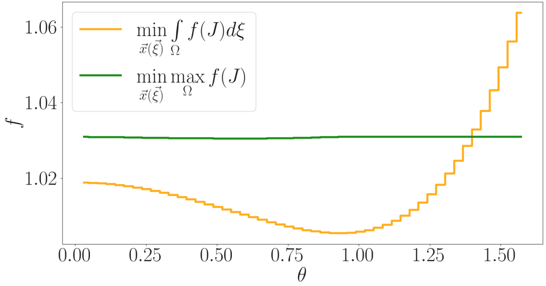


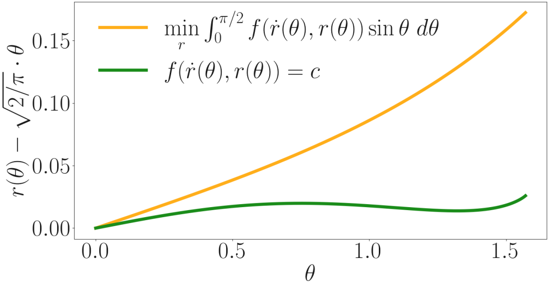


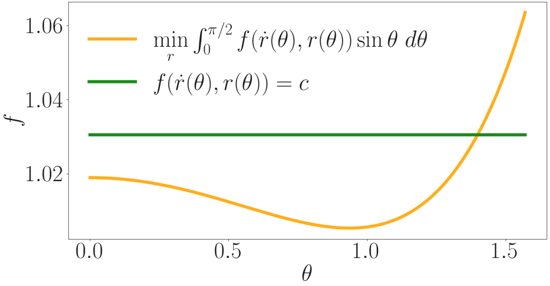


Flat Earth mathematics

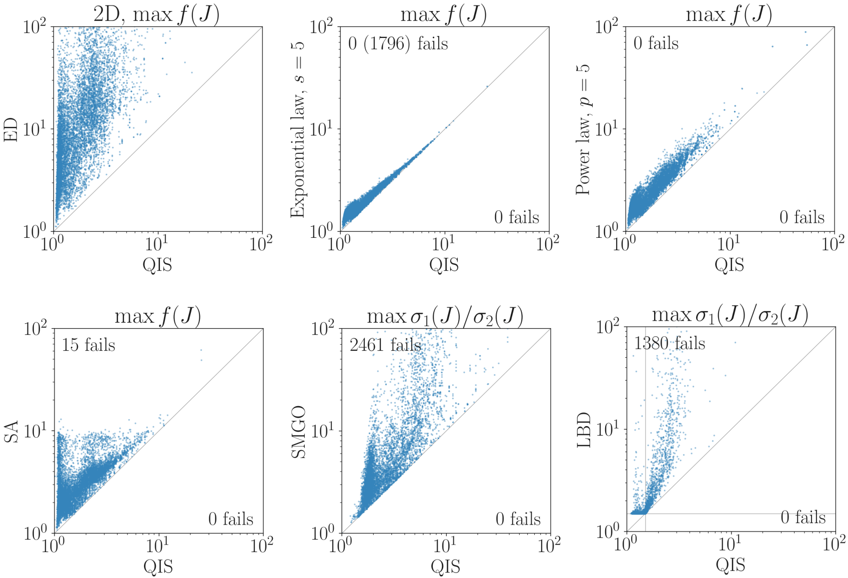




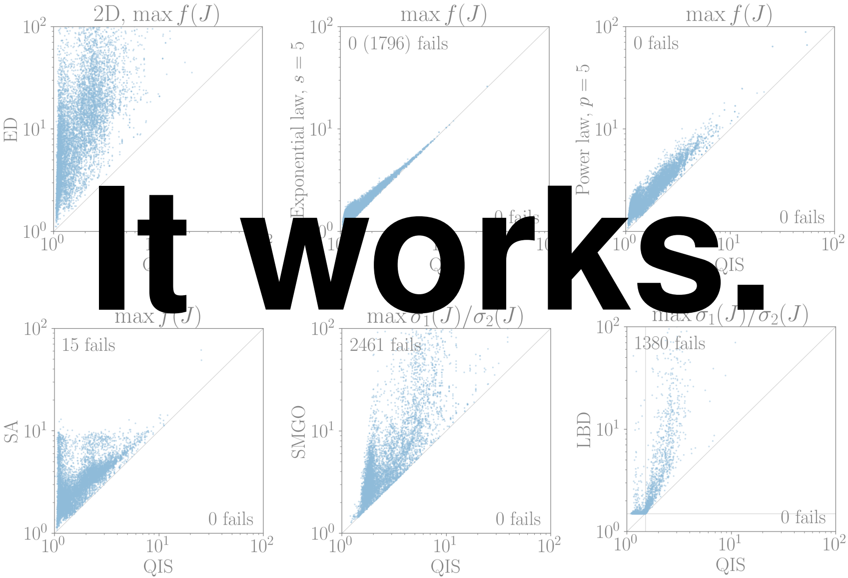




Massive testing



Massive testing



It works.

Lowest distortion mappings

Vladimir Garanzha, Igor Kaporin, Liudmila Kudryavtseva, François Protais, Dmitry Sokolov