High-order solution-based anisotropic mesh adaptation

The log-simplex method

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Doc. ref.:

High-order methods

- Discontinuous Galerkin *Hartmann, Held, Leicht, Prill, Aerospace Sc. Tech. 2010*
- Spectral differences *Liu, Vinokur, Wang, J. Comp. Phys. 2006*
- Residual distributions *Abgrall, Computers and Fluids 2006*

Problematic

- Memory and time costly
- Less efficient if the geometry is not high-ordered approximated

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Mesh adaptation

From a solution $\mathbf{u}$, defined on a domain $\boldsymbol{\Omega} \subset \mathbb{R}^d$, $d = 2, 3$, and a complexity $\mathbf{N}$, find the best mesh $\mathcal{H}$ with complexity $\mathbf{N}$ which minimizes the high-order interpolation error

$$\|\mathbf{u} - \Pi_k \mathbf{u}\|_{L^p(\Omega)},$$

where $p \geq 1$, $k \geq 1$ and Π_k is the projection onto a finite elements space $\mathbb{P}_k$.

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Mesh Adaptation framework

Modify an initial mesh $\mathcal{H}_0$ according to a prescribed size-field

Input

- Initial straight mesh $\mathcal{H}_0$ composed of simplex elements
- $\mathbb{P}_k$ solution u on $\mathcal{H}_0$
- Complexity N

Output

- Adapted straight mesh with prescribed complexity N

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1. Metric-based mesh adaptation
2. From low-order to high-order: log-simplex method
3. Application 1: high-order solution-based mesh adaptation
4. Application 2: curved adapted surface mesh generation

Main idea

Compute geometrical quantities with a custom dot-product

$$\langle x, y \rangle_{\mathcal{M}} = x^t \mathcal{M} y, \quad x, y \in \mathbb{R}^d,$$

with $\mathcal{M}$ a metric $\Rightarrow$ positive definite symmetric matrix of $\mathbb{R}^d$, $d = 2, 3$.

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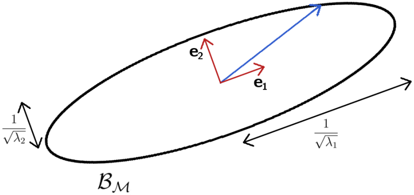
Space is stretched with respect to the features of $\mathcal{M}$.

- Eigenvalues $\{\lambda_1, ..., \lambda_d\}$ of $\mathcal{M}$
  $\Rightarrow$ Anisotropy
- Eigenvectors $\{e_1, ..., e_d\}$ of $\mathcal{M}$
  $\Rightarrow$ Orientation



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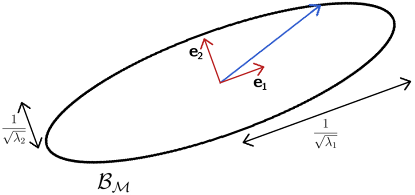

Riemannian metric-space on $\Omega \subset \mathbb{R}^d$

Non-constant $\mathbf{M} = (\mathcal{M}(\mathbf{x}))_{\mathbf{x} \in \Omega}$

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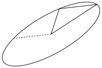
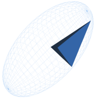
Unit element with respect to $\mathcal{M}$

An element $\mathcal{T}$ is said to be unit with respect to $\mathcal{M}$ if its edges e_i, $i \in \{1, \ldots, d+1\}$ satisfy

$$|e_i|_{\mathcal{M}} = \sqrt{e_i^t \mathcal{M} e_i} = 1.$$





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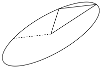
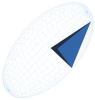
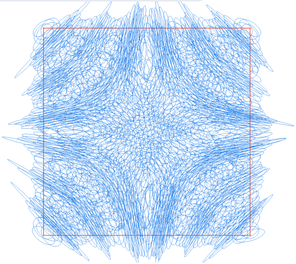


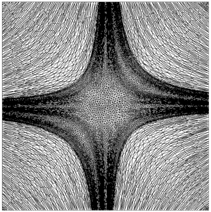


Unit mesh with respect to $\mathbf{M}$

Each element $\mathcal{T}$ of the mesh is unit with respect to $\mathbf{M} = (\mathcal{M}(x))_{x \in \Omega}$.

In practical $\Rightarrow$ quasi unit mesh





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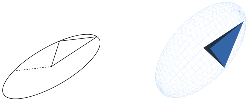

Unit mesh with respect to M

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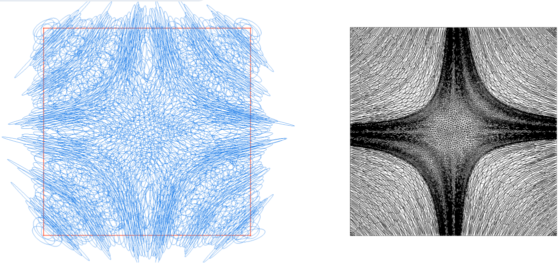

Mesh feature	Metric-space transcription
Length of a segment ab	$\int_0^1 \sqrt{ab^t \mathcal{M}(a + tab) ab}\, dt$
Element area / volume	$C(d)\,(\det(\mathcal{M}))^{-\frac{1}{2}}$
Complexity	$\int_\Omega \sqrt{\det(\mathcal{M}(x))} dx$

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Metric-based mesh adaptation problem

Given a solution u on Ω and an arbitrary polynomial order k, find a metric-space $\mathbf{M_{opt}} = (\mathcal{M}_{opt}(x))_{x\in\Omega}$ with complexity N such that a unit mesh with respect to $\mathbf{M_{opt}}$ minimizes

$$\left\|\mathbf{u} - \Pi_k \mathbf{u}\right\|_{L^p(\Omega)}.$$

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$\mathbb{P}_k$ error estimate

For x_0 a vertex of the mesh $\mathcal{H}$, it comes

$$|(u - \Pi_k u)(x)| \leq |d^{(k+1)}u(x_0)(x - x_0)| + o\left(|x - x_0|_2^{k+1}\right).$$

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$\mathbb{P}_1$ error estimate: Hessian-based methods

For x_0 a vertex of the mesh $\mathcal{H}_0$, it comes

$$|(u - \Pi_1 u)(x)| \leq (x - x_0)^t |H_u(x_0)|(x - x_0) + o\left(|x - x_0|_2^2\right).$$

- Calculus of variations $\Longrightarrow \mathcal{M}_{opt}(x_0) = C(x_0, p, N)|H_u(x_0)|$

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$\mathbb{P}_k$, $k > 1$ error estimate

For x_0 a vertex of the mesh $\mathcal{H}_0$, it comes

$$|(u - \Pi_k u)(x)| \leq |P(x - x_0)| + o\left(|x - x_0|_2^{k+1}\right),$$

with $P = d^{(k+1)}u(x_0)$ a homogeneous polynomial of degree $k+1$.

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For x_0 a vertex of the mesh $\mathcal{H}_0$, it comes

$$|(u - \Pi_k u)(x)| \leq \left((x - x_0)^t \mathcal{Q}(x_0)(x - x_0)\right)^{\frac{k+1}{2}} + o\left(|x - x_0|_2^{k+1}\right).$$

1. Find a metric $\mathcal{Q}(x_0)$ such that $|P(x)| \leq (x^t \mathcal{Q}(x_0) x)^{\frac{k+1}{2}}$

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2. Calculus of variations $\Longrightarrow \mathcal{M}_{opt}(x_0) = C(x_0, p, N)\mathcal{Q}(x_0)$

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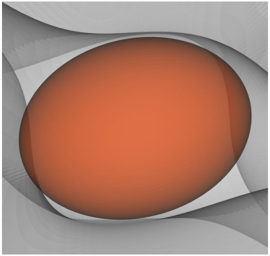
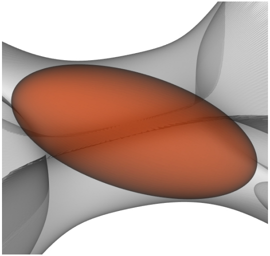
Metric optimization problem

For P a homogeneous polynomial of degree $k+1$, find a metric $\mathcal{Q}$ such that

- $|P(x)| = 1 \implies x^t \mathcal{Q} x \leq 1$
- $\mathcal{B}_{\mathcal{Q}}$ has the largest volume $\Leftrightarrow$ $\det(\mathcal{Q})$ is minimal





Non-linear problem !

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Principle

1. Instead of $\mathcal{Q}$, consider $\mathcal{L} = \log(\mathcal{Q})$
    - $\text{trace}(\mathcal{L})$ has to be minimized (since $\det(\mathcal{Q}) = \exp(\text{trace}(\mathcal{L}))$)

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     $\Rightarrow$ Introduce non-linear constraints on $\mathcal{L}$

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Log matrix optimization problem

For P a homogeneous polynomial of degree $k+1$, find a symmetric matrix $\mathcal{L}$ such that

- $|P(x)| = 1 \quad \Longrightarrow \quad x^t \mathcal{L} x \geq -|x|_2^2 \log(|x|_2^2)$
- $\text{trace}(\mathcal{L})$ is minimal

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Advantages

1. Cost function and constraints are linear
2. Only consider a finite set of points $\{x_1, ..., x_n\}$

$\Rightarrow$ Solved by a classical simplex algorithm

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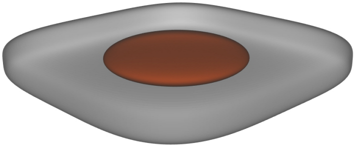
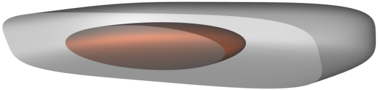
How to recover the initial constraints ?





Iterative process

1. Choose a set $\{x_1, ...x_n\} \in \mathbb{R}^d$ such that $|P(x_i)| = 1$
2. Compute the optimal symmetric matrix $\mathcal{L}$ and its associated metric $\mathcal{Q}$
3. Perform the change of variable $y = \mathcal{Q}^{\frac{1}{2}} x$
   $\Rightarrow$ replace P by $P \circ \mathcal{Q}^{-\frac{1}{2}}$

If $\mathcal{Q}$ converges, the log constraints and the initial ones are **equivalent**.

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High-order mesh adaptation workflow

Mesh adaptation schedule

1. Compute the high-order solution u
2. Compute the high-order differential $d^{(k+1)}u$
3. Compute the metric $\mathcal{Q}$ approximating $d^{(k+1)}u$
4. Derive the optimal metric-field $\mathcal{M} = C(k, N)\mathcal{Q}$ such that $\mathcal{C}(\mathcal{M}) = N$
5. Adapt the mesh

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Main features

- Global control of the interpolation error
- Every scale of the solution is caught
- Arbitrary order k
- Any kind of polynomial interpolations
- Consistent with the Hessian-based methods if $k = 1$

High-order solver: Argo

Discontinuous Galerkin high-order multiphysics solver

Applications

- Turbomachinery
- External flow
- Ablation
- Additive Manufacturing
- Thermodynamics

Mesh adapter: MAdLib

Open source mesh-adaptation library

Functionalities

- Metric-based mesh-adaptation
- High-order solution-based mesh adaptation
- High-order surface mesh generation
- Level-set conforming mesh generation

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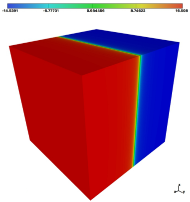
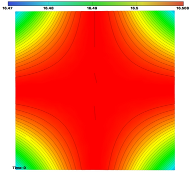
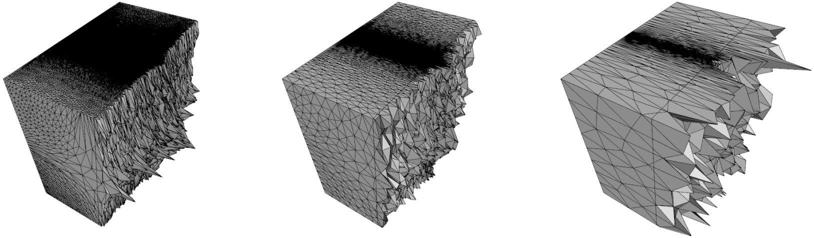
- $\Omega = \left[-\frac{1}{2},\ \frac{1}{2}\right]^3$
- $u(x, y, z) = 10 \arctan(100x) + \cos(yz)$





$\mathbb{P}_1$, $\mathbb{P}_2$ and $\mathbb{P}_3$ adapted meshes (around $350k$ degrees of freedom).



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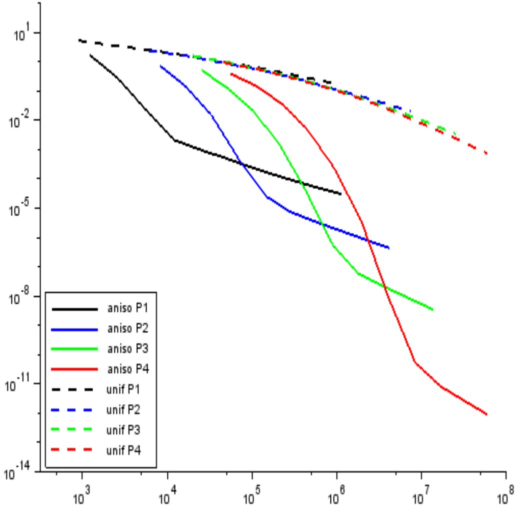
Interpolation error with respect to number of dofs



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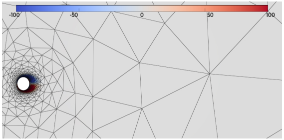
Flow features

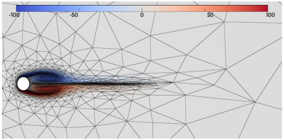
- Compressible flow
- Reynolds 185

Adaptation features

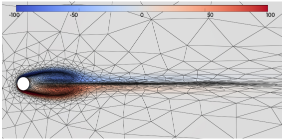
- Adapted to the $\mathbb{P}_2$ vorticity
- Adapted every 5 time steps



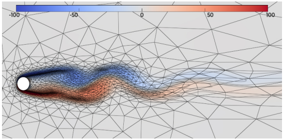
(a) 2.5e-02s



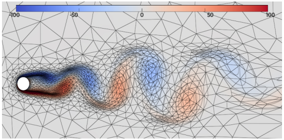
(b) 3.5e-01s



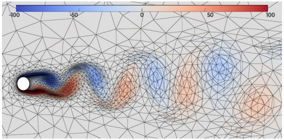
(c) 6.5e-01s



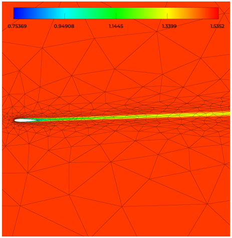
(d) 1.2s



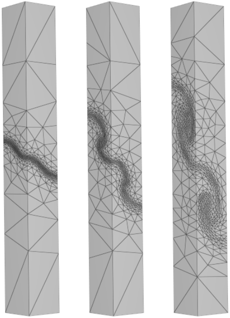
(e) 1.5s



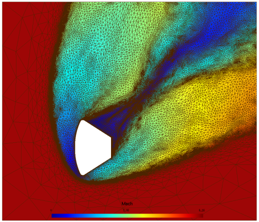
(f) 1.7s



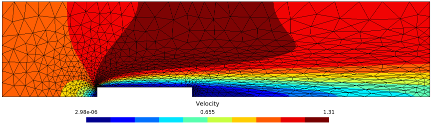
$\mathbb{P}_4$ **flow on a Naca12 geometry**



$\mathbb{P}_2$ **Rayleigh-Taylor instability**



$\mathbb{P}_1$ **hypersonic flow**



$\mathbb{P}_2$ **flow around a step**

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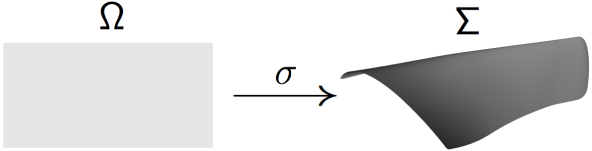
Surface parameterization



Purpose

Input

- CAD model
- $\mathbb{P}_1$ initial mesh $\mathcal{H}_0$
- Complexity N

Output

- Adapted curved mesh $\mathcal{H}^k$ with prescribed complexity N

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High-order surface mesh generation

Tangent plane $\mathcal{P}_{x_0}$ in $x_0 = \sigma(u_0, v_0) \in \Sigma$.

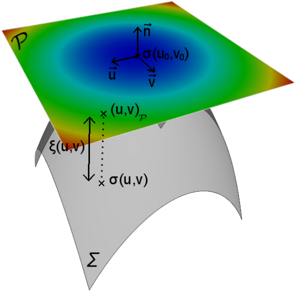
- Local coordinates system $\mathcal{B}_\mathcal{P} = (x_0, \vec{u}, \vec{v})$:
- Normal vector $\vec{n} = \vec{u} \wedge \vec{v}$.

Local "gap" function on $\mathcal{P}_{x_0}$

$$\begin{array}{ccc} \mathcal{U} \subset \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ (u, v)_\mathcal{P} & \longmapsto & \xi((u, v)_\mathcal{P}) \end{array}$$

Gap between the surface and its tangent plane

In the normal direction



Principle

1. At each vertex x of $\mathcal{H}_0$, build a high-order representation of ξ
2. Use the log-simplex method to get a 2D optimal metric on $\mathcal{P}_x$
3. Map the 2D metric to the 3D physical space
4. Adapt and curve the mesh

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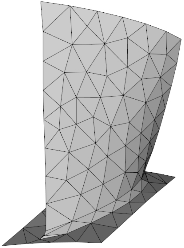
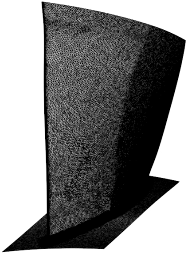
Initial linear mesh

- 149 vertices
- 276 triangles



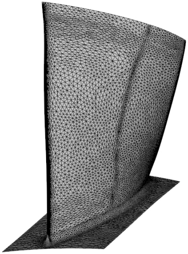
Linear adapted mesh

- 65800 vertices
- 131176 triangles



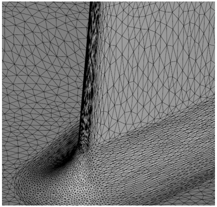
$\mathbb{P}_2$ adapted mesh

- 65254 dof
- 32551 $\mathbb{P}_2$ triangles



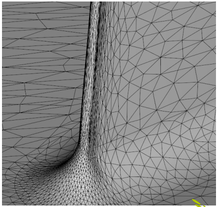
$\mathbb{P}_1$ mesh

Zoom



$\mathbb{P}_2$ mesh

Zoom



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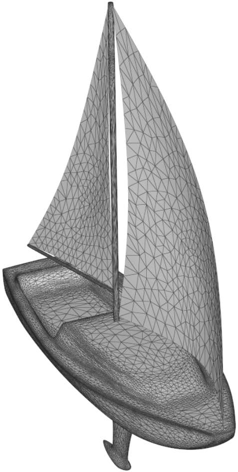
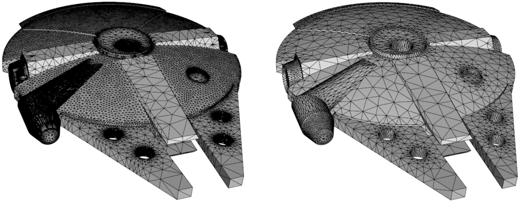
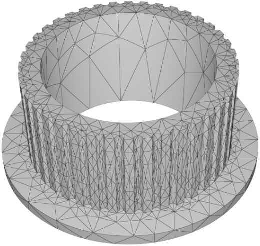


$\mathbb{P}_3$ **sail boat**



$\mathbb{P}_1$ **vs** $\mathbb{P}_2$ **millenium falcon**



$\mathbb{P}_3$ **coarse Fender bearing**

Conclusions & Perspectives

Achievements

- High-order solution-based mesh adaptation
- Curved surface-based mesh adaptation

Limitations

- Smooth solution
- Fixed interpolation order

Perspectives

- H-p adaptation
- Curve volume mesh

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Thank you !